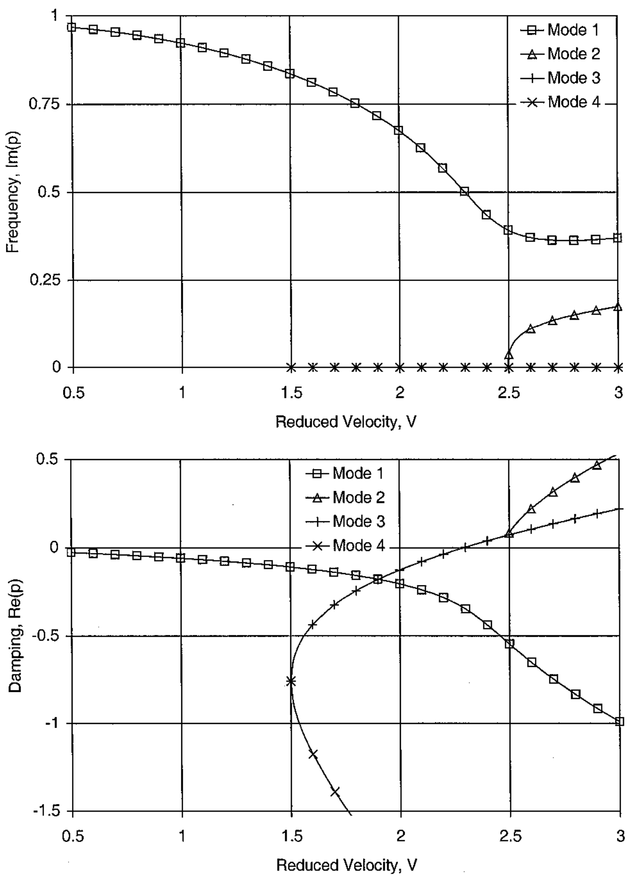


**Fig. 3 Solution of the flutter equation with the aerodynamic moment approximated by a first-order expansion from the imaginary axis.**

three additional roots appear, one oscillatory and two nonoscillatory. The additional oscillatory mode is stable, and its frequency increases to match that of the oscillatory mode at $V = 2.26$. The two nonoscillatory modes are initially both stable. One becomes more stable and the other becomes less stable, becoming unstable at the divergence speed.

In the $g$ method of Chen[6] the aerodynamic coefficient is approximated by

$$C_{M\alpha}(p/V, a) \approx C_{M\alpha}(ik, a) + (p/V - ik)\frac{\partial C_{M\alpha}(ik, a)}{\partial ik} \tag{7}$$

and leads to the equation of motion

$$p^2 - \frac{V}{2\pi\mu r_\alpha^2}\frac{\partial C_{M\alpha}(ik, a)}{\partial ik}p + 1 - \frac{V^2}{2\pi\mu r_\alpha^2}\left(C_{M\alpha}(ik, a) - ik\frac{\partial C_{M\alpha}(ik, a)}{\partial ik}\right) = 0 \tag{8}$$

The solution is shown in Fig. 3. The pitch mode exists over the entire speed range, and its frequency decreases but does not go to zero. At $V = 1.50$, two stable nonoscillatory roots appear. One becomes more stable and the other becomes less stable, becoming unstable at the divergence speed. At $V = 2.49$, an unstable oscillatory root bifurcates from the divergence root.

## Conclusions

The present study shows that divergence is predicted very differently by three forms of the $p$–$k$ flutter equation even for the simplest of cases. The differences between the results are entirely due to different approximations to the generalized aerodynamic forces. All three forms do, however, predict the same divergence speed. This is to be expected because divergence occurs at $p = 0$, where the three approximations to the pitching moment coefficient are equivalent. However, aeroelastic divergence of free-flying aircraft does not occur at $p = 0$, and we can expect the three different forms to predict different divergence speeds. None of the forms predict that the frequency of the pitch mode goes to zero at divergence, which is in agreement with the analysis in Ref. 7.

## References

[1]Cole, S. R., "Divergence Study of a High-Aspect-Ratio, Forward Swept Wing," *Journal of Aircraft*, Vol. 25, No. 5, 1988, pp. 478–480.

[2]Rodden, W. P., and Bellinger, E. D., "Aerodynamic Lag Functions, Divergence, and the British Flutter Method," *Journal of Aircraft*, Vol. 19, No. 7, 1982, pp. 596–598.

[3]Rodden, W. P., and Bellinger, E. D., "Unrestrained Aeroelastic Divergence in a Dynamic Stability Analysis," *Journal of Aircraft*, Vol. 19, No. 9, 1982, pp. 796, 797.

[4]Hassig, H. J., "An Approximate True Damping Solution of the Flutter Equation by Determinant Iteration," *Journal of Aircraft*, Vol. 8, No. 11, 1971, pp. 885–889.

[5]Rodden, W. P., Harder, R. L., and Bellinger, E. D., "Aeroelastic Addition to NASTRAN," NASA CR 3094, 1979.

[6]Chen, P. C., "A Damping Perturbation Method for Flutter Solution: The g-Method," *Proceedings of the International Forum on Aeroelasticity and Structural Dynamics*, Pt. 1, CP-1999-209136/PT1, NASA, 1999, pp. 433–441.

[7]Heeg, J., and Dowell, E. H., "Aerodynamic and Aeroelastic Insights Using Eigenanalysis," *Proceedings of the International Forum on Aeroelasticity and Structural Dynamics*, Pt. 1, CP-1999-209136/PT1, NASA, 1999, pp. 403–413.

[8]Von Kármán, T., and Sears, W. R., "Airfoil Theory for Non-Uniform Motion," *Journal of the Aeronautical Sciences*, Vol. 5, No. 10, 1938, pp. 379–390.

[9]Jones, W. P., "Aerodynamic Forces on Wings in Non-Uniform Motion," Repts. and Memoranda 2117, 1945, Aeronautical Research Council, London.

# Aeroelastic Divergence and Aerodynamic Lag Roots

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## Nomenclature

| | | |
|---|---|---|
| $B$ | = | damping matrix |
| $c$ | = | airfoil chord or mean wing chord |
| $g$ | = | structural damping coefficient, also $2\,\mathrm{Re}(p)c/[\ln(2)V]$ |
| $K$ | = | stiffness matrix |
| $k$ | = | reduced frequency, $\omega c/2V$ |
| $M$ | = | inertia matrix |
| $m$ | = | airfoil mass per unit span |
| $p$ | = | eigenvalue, $\omega(\gamma \pm i)$ |
| $Q^I$ | = | aerodynamic damping matrix, $\mathrm{Im}[Q(k)]$ |
| $Q^R$ | = | aerodynamic stiffness matrix, $\mathrm{Re}[Q(k)]$ |
| $Q(k)$ | = | matrix of generalized aerodynamic forces |
| $\{u\}$ | = | vector of degrees of freedom |
| $V$ | = | true airspeed |
| $\gamma$ | = | decay rate coefficient |
| $\mu$ | = | airfoil mass ratio $m/[\pi\rho(c/2)^2]$ |
| $\rho$ | = | air density |
| $\omega$ | = | angular frequency, rad/s |

## Introduction

FLUTTER analysis methods have been used to predict aeroelastic divergence for a variety of cases, including a hypothetical jet transport wing known as the BAH wing[1–3] and an airfoil with two[4]

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and three[5] degrees of freedom. In the most of these cases, it was concluded that divergence was due to an aerodynamic lag root becoming unstable. In the case of the BAH wing, the conclusions have not been consistent. In Ref. 1, which used aerodynamic strip theory and a transient flutter method, and Ref. 2, which used aerodynamic lifting surface theory and a $p$–$k$ flutter method, it was concluded that divergence was due to an aerodynamic lag root becoming unstable. In Ref. 3, which used aerodynamic lifting surface theory and a $p$–$k$ flutter method, it was concluded that a structural mode diverges. Some of the confusion may be because the mode tracking procedure inherent in the $p$–$k$ flutter method does not provide a complete solution inasmuch as it has no means of detecting additional roots and does not converge for all roots. In the present study, a more complete solution to Rodden, Harder, and Bellinger's form of the $p$–$k$ flutter equation[6] for the two-degree-of-freedom airfoil[4] was obtained by a root search procedure similar to that proposed by Chen.[7]

## Flutter Equation

Rodden, Harder, and Bellinger's form of the $p$–$k$ flutter equation[6] is

$$\left[Mp^2 + \left(B - \tfrac{1}{4}\rho c V Q^I/k\right)p + \left(K - \tfrac{1}{2}\rho V^2 Q^R\right)\right]\{u\} = 0 \quad (1)$$

All of the coefficient matrices in the flutter equation are real; therefore, the eigenvalues are either real or occur in complex conjugate pairs.

An iterative solution procedure is normally used because the aerodynamic matrix depends on the imaginary part of $p$. A value for $k$ is assumed, either based on the converged solution at a previous speed or on the natural frequency of the mode under study. The eigenvalue problem is solved, and the eigenvalue corresponding to the mode under study is identified. The frequency of the eigenvalue is compared to the assumed value of $k$. If the difference exceeds a predetermined value a new value for $k$ is calculated from the frequency of the eigenvalue, and the process is repeated. This is known as the method of successive approximation[8] and converges only if the absolute value of the slope of $\omega$ with respect to $k$ is less than $2V/c$, that is, if the magnitude of the slope of $\omega$ with respect to $2kV/c$ is less than 1.

## Examples

The two examples of an airfoil with pitch and plunge degrees of freedom of Ref. 4 were analyzed. The characteristics of the two examples are the same except for the center of gravity location: In example 1, it is at 37% chord, and in example 2, it is at 45% chord. The elastic axis is at 40% chord, the radius of gyration about the elastic axis is 25% of the chord, and the mass ratio $\mu = 20.0$. The uncoupled bending and torsion frequencies are 10.0 and 25.0 rad/s, respectively, with equal structural damping coefficients $g = 0.03$ in both modes. Except that the flutter speed of example 1 is above the divergence speed and the flutter speed of example 2 is below the divergence speed, the results are qualitatively very similar. Only the results for example 1 are presented and discussed.

## Solution Method

Incompressible flow was assumed and the two-term approximation of Jones[9] to the Theodorsen circulation function was used in the calculation of the generalized aerodynamic forces.

At each velocity over the range 0.5–300 ft/s in increments of 0.5 ft/s, the eigenvalue problem was solved for a range of $k$ values. Eigenvalues would be valid roots of the flutter equation if the imaginary part of the eigenvalue corresponded to the assumed $k$ value. A change of sign of the difference [Im($p$) − $2kV/c$] indicated the presence of a root, and the value of the root was determined by linear interpolation. The eigenvalues of the flutter equation for example 1 at 175 ft/s are plotted against $2kV/c$ in Fig. 1. Only eigenvalues with nonnegative imaginary parts are shown. The first eigenvalue is complex over the entire range of frequencies, with one root at 19.0 rad/s. The second and third eigenvalues are real up to 2.5 rad/s with both having a root at 0.0 rad/s. Above 2.5 rad/s, the second and third eigenvalues form a complex conjugate pair, with the third eigenvalue having a negative imaginary part. The second eigenvalue has two more roots, at 2.7 and at 8.4 rad/s.

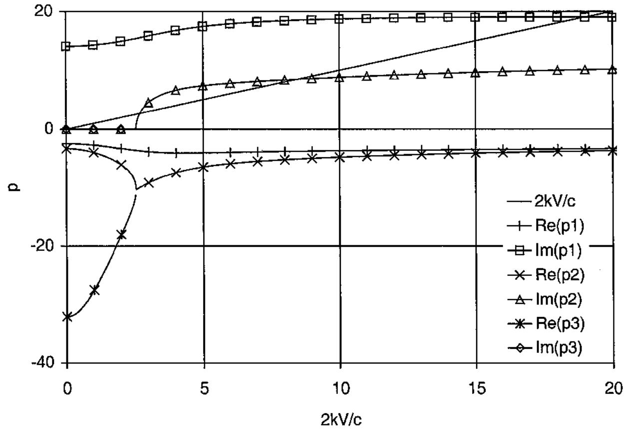


**Fig. 1 Eigenvalues of example 1 at 175 ft/s plotted against $2kV/c$.**

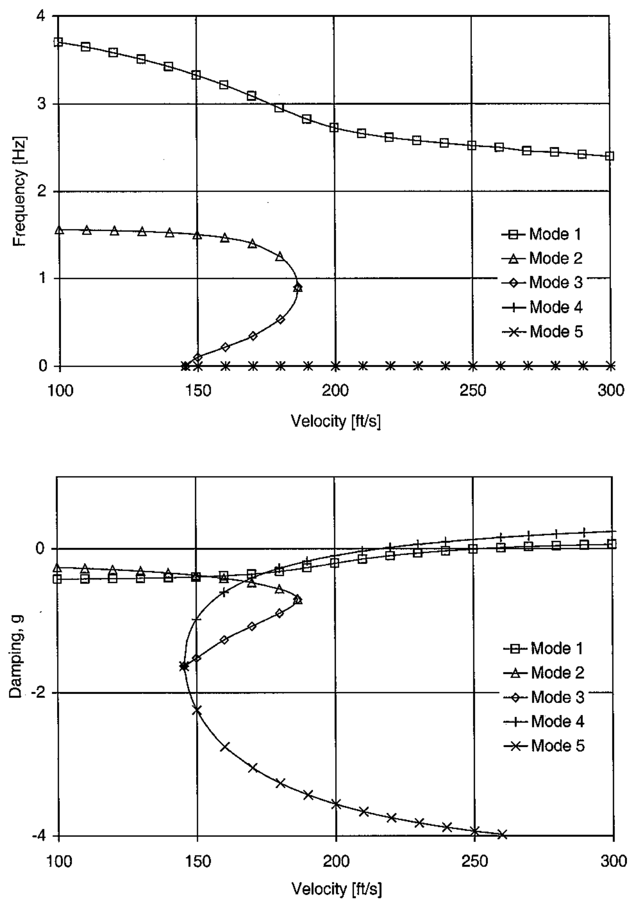


**Fig. 2 Solution of example 1.**

## Results

The $V$–$f$ and $V$–$g$ plots for example 1 are shown in Fig. 2. The damping values were all normalized as defined in Ref. 3 for nonoscillatory roots. This was necessary to show a smooth transition from a complex to a real root. The pitch mode, mode 1, flutters at 253.0 ft/s.

The plunge mode, mode 2, exists up to a speed of 186.5 ft/s. Toward the end of this range, the frequency decreases sharply. It then meets up with an additional oscillatory mode, mode 3. The frequency of mode 3 decreases further with reducing speed, becoming zero at a speed of 145.5 ft/s. Note that mode 3 has a corresponding complex conjugate eigenvalue that is not a valid root because its frequency does not match the assumed $k$ value. As the frequency of mode 3 reaches zero, the complex conjugate eigenvalue becomes a valid root. As the speed increases from 145.5 ft/s, one real root, mode 4, becomes less stable and becomes unstable at the divergence speed, 216.5 ft/s. The other real root, mode 5, becomes more stable with increasing speed.

## Discussion

It can be seen from Fig. 1 that the method of successive approximation would not converge to the second root of the second eigenvalue. The plot of imaginary part of the second eigenvalue $p_2$ crosses the line $2kV/c$ from below, implying that the slope of $\mathrm{Im}(p_2)$ with respect to $2kV/c$ is greater than 1. Note that a Newton–Raphson solution method[8] would still converge.

The plot of the imaginary part of the second eigenvalue (Fig. 1) retains its shape, but moves to the right with increasing speed and to the left with decreasing speed. This is consistent with the appearance of the two real roots and one oscillatory root at 145.5 ft/s and the disappearance of the two oscillatory roots at 186.5 ft/s. Note that the shape of the plot is peculiar to Rodden, Harder, and Bellinger's form[6] of the $p$–$k$ flutter equation.

From Fig. 2, it can be seen that a mode tracking procedure would track the plunge mode up to the speed where the mode ceases to exist or turns around and then converge to one of the real roots. This is similar to the behavior reported in Ref. 4, except that the solution converged to the real root at speeds where the oscillatory root should still exist.

## Summary

Whenever the frequency of a structural mode goes to zero, one would expect the complex root to be replaced by two real roots at higher speeds. A mode tracking procedure would track only one of the real roots, implying that the solution would be incomplete. The divergence roots in the case of the two examples of Ref. 4 are the logical continuation of structural modes after their frequencies have gone to zero. Calling them aerodynamic lag roots does not seem justified.

## References

[1]Rodden, W. P., and Stahl, B., "A Strip Method for Prediction of Damping in Subsonic Wind Tunnel and Flight Flutter Tests," *Journal of Aircraft*, Vol. 6, No. 1, 1969, pp. 9–17.

[2]Rodden, W. P., *MSC/NASTRAN Handbook for Aeroelastic Analysis*, Vol. 2, MacNeal–Schwendler Corp., Los Angeles, CA, 1987, pp. 6.2-22–6.2-26.

[3]Rodden, W. P., and Johnson, E. H., *MSC/NASTRAN Version 68 Aeroelastic Analysis User's Guide*, MacNeal–Schwendler Corp., Los Angeles, CA, 1994, pp. 516–523.

[4]Rodden, W. P., and Bellinger, E. D., "Aerodynamic Lag Functions, Divergence, and the British Flutter Method," *Journal of Aircraft*, Vol. 19, No. 7, 1982, pp. 596–598.

[5]Rodden, W. P., and Bellinger, E. D., "Unrestrained Aeroelastic Divergence in a Dynamic Stability Analysis," *Journal of Aircraft*, Vol. 19, No. 9, 1982, pp. 796, 797.

[6]Rodden, W. P., Harder, R. L., and Bellinger, E. D., "Aeroelastic Addition to NASTRAN," NASA CR 3094, 1979.

[7]Chen, P. C., "A Damping Perturbation Method for Flutter Solution: The g-Method," *Proceedings of the International Forum on Aeroelasticity and Structural Dynamics*, Pt. 1, CP-1999-209136/PT1, NASA, 1999, pp. 433–441.

[8]Dorn, W. S., and McCracken, D. D., *Numerical Methods with Fortran IV Case Studies*, Wiley, New York, 1972, pp. 17–22, 25–27.

[9]Jones, W. P., "Aerodynamic Forces on Wings in Non-Uniform Motion," Repts. and Memoranda 2117, 1945, Aeronautical Research Council, London.

# Unrestrained Aeroelastic Divergence and the *p*–*k* Flutter Equation

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## Nomenclature

| | | |
|---|---|---|
| $B$ | = | viscous structural damping matrix |
| $c$ | = | airfoil chord |
| $g$ | = | structural damping coefficient, also $2\mathrm{Re}(p)c/[\ln(2)V]$ |
| $K$ | = | stiffness matrix |
| $k$ | = | reduced frequency, $\omega c/2V$ |
| $M$ | = | inertia matrix |
| $m$ | = | airfoil mass per unit span |
| $p$ | = | differential operator, $\mathrm{d}/\mathrm{d}t$ |
| $Q$ | = | matrix of generalized aerodynamic forces, a function of Mach number and $pc/2V$ |
| $Q^I$ | = | imaginary part of $Q$ |
| $Q^R$ | = | real part of $Q$ |
| $\{u\}$ | = | vector of degrees of freedom |
| $V$ | = | true airspeed |
| $\mu$ | = | airfoil mass ratio $m/[\pi\rho(c/2)^2]$ |
| $\rho$ | = | air density |
| $\omega$ | = | angular frequency, $\mathrm{Im}(p)$ |

## Introduction

In Ref. 1 the aeroelastic divergence of two unrestrained airfoil–body systems was investigated using the British ($p$–$k$) flutter method. It was concluded that the systems diverged in an oscillatory fashion rather than quasi statically as is the case with restrained systems. It was also found that the divergence speeds as determined from the flutter analysis were slightly different from those obtained by quasi-static unrestrained divergence analysis.[2]

In the present study, the second example of Ref. 1 was analyzed using four forms of the $p$–$k$ flutter equation, namely, Hassig's form,[3] Hassig's form with the rigid plunge displacement degree of freedom eliminated, Rodden, Harder, and Bellinger's form,[4] and the exact equation of motion. These results show the link between quasi-static unrestrained divergence analysis and the dynamic stability methods.

## Solutions

The characteristics of the two examples of Ref. 1 are the same except for the center of gravity location: In example 1 it is at 37% chord and in example 2 it is at 45% chord. The chord is 6 ft, the elastic axis is at 40% chord, the radius of gyration about the elastic axis is 25% of the chord, and the mass ratio $\mu = 20.0$. The uncoupled bending and torsion frequencies are 10.0 and 25.0 rad/s, respectively, with equal structural damping coefficients $g = 0.03$ in both modes. The airfoil plunge spring is attached to a body with only a plunge degree of freedom and mass equal to the airfoil mass.

Incompressible flow was assumed, and Jones's approximation[5] to the Theodorsen circulation function was used in the calculation of the aerodynamic coefficients. A root search technique[6] was used rather than the traditional mode tracking method of solution. The damping values were all normalized as defined in Ref. 7 for nonoscillatory roots. This was necessary to show smooth transitions from complex to real roots.

The equation of motion of the system is

$$\left[Mp^2 + Bp + K - \tfrac{1}{2}\rho V^2 Q(pc/2V)\right]\{u\} = 0 \qquad (1)$$

For most practical problems, the solution of Eq. (1) is a formidable task, mainly because of the dependence of the generalized forces

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